J. Fluid Mech. (1980), *vol.* 96, *part* 1, *pp.* 81–104
Printed in Great Britain

Asymptotic analysis of plane turbulent Couette–Poiseuille flows

By KURT O. LUND AND WILLIAM B. BUSH
University of California, San Diego, California

(Received 12 June 1978 and in revised form 8 December 1978)

Matched asymptotic expansions are used to describe turbulent Couette–Poiseuille flow (plane duct flow with a pressure gradient and a moving wall). A special modification of conventional eddy-diffusivity closure accounts for the experimentally observed non-coincidence of the locations of zero shear stress and maximum velocity. An asymptotic solution is presented which is valid as the Reynolds number tends to infinity for the whole family of Couette–Poiseuille flows (adverse, favourable, and zero pressure gradients in combination with a moving wall). It is shown that plane Poiseuille flow is a limiting case of Couette–Poiseuille flow. The solution agrees with experimental data for plane Couette flow, for the limiting plane Poiseuille flow, and for a special case having zero net flow and an adverse pressure gradient. The asymptotic analysis shows that conventional eddy diffusivity closures are inadequate for general Couette–Poiseuille flows.

1. Introduction

This paper studies the macroscopic structure of plane turbulent incompressible Couette–Poiseuille flows by means of limit process expansion techniques. The formulation is in terms of the conventional Reynolds time-averaged equations, subject to an eddy-viscosity closure postulate relating the Reynolds stress to the mean flow quantities. Such an approach has been employed to obtain solutions for plane turbulent Poiseuille flow (Yajnik 1970; Bush & Fendell 1972, 1974; Fendell 1972), i.e. flow in an infinitely long channel with plane stationary parallel walls, driven by an imposed constant favourable pressure gradient. Here, based on the methods developed for Poiseuille flow, solutions are obtained for plane turbulent Couette flow, i.e. flow between parallel surfaces moving with respect to each other, with no imposed pressure gradient; and, more generally, for plane turbulent Couette–Poiseuille flow, i.e. flow between parallel surfaces moving with respect to each other, and having an imposed constant favourable or adverse pressure gradient. In this paper, it is shown that Poiseuille flow is a limiting case of Couette–Poiseuille flow.

Poiseuille flow has been investigated intensely, both experimentally and theoretically, and it has remained of interest to the present time (cf., e.g., Hussain & Reynolds 1975). Recently, a number of higher-order closure models has been introduced (cf. Reynolds 1976) for the purpose of extending to other flows the results obtained for turbulent Poiseuille flow. These models have had only limited success, and none of them predicts duct flows as well as the eddy-viscosity closure. In the present work, the asymptotic description of the entire family of Couette–Poiseuille flows is sought

0022-1120/80/4309-7490 $02.00

as the Reynolds number (based on the moving wall speed and the plane walls separation distance) tends to infinity. In accordance with the above remarks, only the simplest of closure hypotheses, which includes the essential physics of the flow, is advanced.

Since the original investigation by Couette (1890), which was for circular geometry, Couette flow has remained of interest to researchers in fluid mechanics. Burgers (1922) and Heisenberg (1922) considered the nature of plane turbulent Couette flow, and von Kármán (1937) calculated this flow using his similarity theory (cf. Robertson 1959). However, because of the obvious difficulties in achieving fully-developed turbulent flow in a long channel with a moving wall, plane Couette flow has not been investigated experimentally until recent times. Reichardt (1959) and Robertson (1959) provided the first comprehensive measurements of averaged flow quantities in air, water and oil. The measurements of Chue (1969) are for plane Couette flow in a polymer solution; while Robertson & Johnson (1970) included (some) turbulence quantities in their measurements. Leutheusser & Chu (1971) used flowing water in a hydraulic flume as the moving boundary for the air above the flume to investigate the transition and low-Reynolds-number regimes of plane Couette flow. These various experiments have produced similar results, but with significant differences in the friction coefficients deduced from the measurements. The present theory results in a resistance law which lies approximately midway between the extremes of the data, as shown in § 4.1.

Plane Couette flow has also been investigated theoretically over the past twenty years. Reichardt (1956, 1959) used a quadratic eddy-diffusivity distribution in conjunction with his sublayer damping function, but found it necessary to develop two separate models to account for the high- and low-Reynolds-number data. Robertson (1959) used a mixing-length model to empirically fit his data. Szablewski (1968) used an exponential mixing-length model in an attempt at a unified description of plane Couette, plane Poiseuille, and pipe flows, and obtained a piecewise-continuous solution, which agrees with Reichardt's high-Reynolds-number velocity measurements. Korkegi & Briggs (1968, 1970) calculated compressible Couette flow, using a mixing-length formulation, and compared their results with Robertson's data. Chue & McDonald (1970) used a piecewise-continuous eddy-diffusivity function to fit Chue's data, and Leutheusser & Chu (1971) introduced three separate empirical correlations to fit their resistance data. Hoffmeister (1976) used a modified similarity theory to fit part of the data of Reichardt. All of the above analyses adjust various parameters to fit the experimental data, and none of them is valid over the full range of Reynolds numbers for turbulent flow. In the present asymptotic analysis, a solution is obtained which is valid over the full range of Reynolds numbers and which does not rely upon fitting to the Couette flow skin friction data.

Because of its importance to lubrication theory, efforts have been directed recently to the calculation of turbulent Couette–Poiseuille flows, but experimental data for these flows are practically non-existent. Constantinescu (1959) introduced the use of Prandtl's mixing length to turbulent lubrication theory; subsequent modifications to Constantinescu's analysis were presented by Ng (1964). More recent analyses have involved numerical integration using different eddy-viscosity models for different regions of the flow (cf. Elrod & Ng 1967), and a version of Prandtl's turbulence energy model, i.e. a (so-called) one-equation model of turbulence (cf. Ho & Vohr 1974).

Although the results of these two cited analyses are found to be in mutual agreement, it should be noted that both analyses are based on the conventional Reynolds stress–eddy-viscosity relationship, which does not take into account the possibility that the locations of the maximum velocity and zero Reynolds stress may not coincide. This non-coincidence has been established experimentally for duct flows with asymmetric boundary conditions (cf. Hanjalić & Launder 1972; Rehme 1974).

In the present paper, an essential modification, which allows for the treatment of Couette–Poiseuille flows having this non-coincidence, is introduced to a simple, but continuous, eddy-viscosity closure model. By means of this modification, it is possible to develop an asymptotic theory for the whole family of Couette–Poiseuille flows.

In § 2, the equations of motion for plane turbulent Couette–Poiseuille flow are developed as a singular perturbation eigenvalue problem, with the (dimensionless) friction velocity appearing as the eigenvalue to be determined. The asymptotic solutions of the problem as the Reynolds number (based on the moving wall speed and the plane walls separation distance) tends to infinity, with the ratio of the shear stresses at the two walls held fixed, are presented in § 3. The matching requirements near the two walls result in a friction velocity and a friction coefficient of the same asymptotic forms as previously found for plane Poiseuille flow (cf. Bush & Fendell 1972). Through an examination of the higher-order contributions of the asymptotic solutions, it is shown that the positions of maximum velocity and zero Reynolds stress do not coincide for the range of Couette–Poiseuille flows that has a local maximum in the velocity field. In § 4, there is a comparison of the asymptotic theory results with available experimental data for plane Couette flow, the limiting case of plane Poiseuille flow, and for a special case of Couette–Poiseuille flow, in which the ends of the duct are blocked creating a zero net flow, adverse pressure gradient condition (cf. Huey & Williamson 1974). The asymptotic theory is found capable of producing agreement with all of these data over the full range of Reynolds numbers investigated.

2. Formulation of the problem

2.1. *The equations of motion*

Consider the steady, two-dimensional, fully-developed turbulent Couette–Poiseuille flow of a fluid of constant density and viscosity ($\rho^*, \nu^* =$ consts.) between two parallel, plane, smooth walls of infinite axial extent, with the lower wall fixed and the upper wall moving at a constant speed (see figure 1).† Let

$$x^* = h^* x, \quad y^* = h^* y \tag{2.1}$$

represent the co-ordinates tangential and normal to the fixed lower wall, with h^* the normal distance between the walls. The mean velocity components in the x^* and y^* directions, and the applied mean axial pressure gradient are

$$u^* = u^*(y^*; \ldots) = V^* u(y; \ldots), \quad v^* \equiv 0, \tag{2.2}$$

$$P^* = P^*(\ldots) = -\frac{dp^*}{dx^*} = \frac{\rho^* V^{*2}}{h^*} P(\ldots) = \text{const.}, \tag{2.3}$$

† Here, all dimensional variables are denoted by a superscript asterisk.

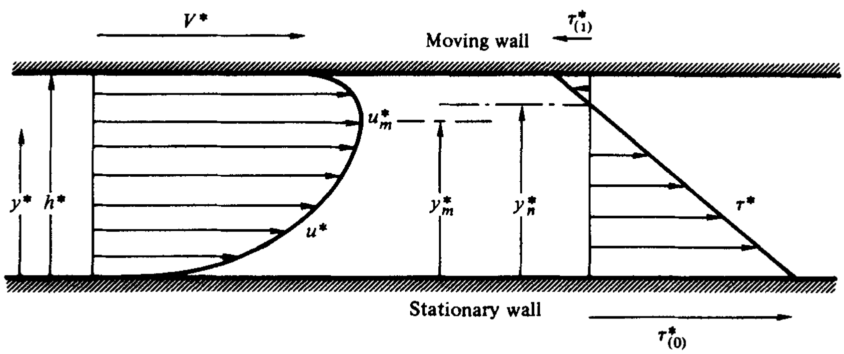

FIGURE 1. Definition sketch for Couette–Poiseuille flow.

where V^* is the constant speed of the moving upper wall. The relevant Newton laminar stress and Reynolds turbulent stress, respectively, are

$$\begin{aligned} \tau_N^* &= \tau_N^*(y^*; \dots) = \rho^* \nu^* \frac{du^*}{dy^*} \\ &= \frac{\rho^* V^{*2}}{(V^* h^*/\nu^*)} \tau_N(y; \dots) = \frac{\rho^* V^{*2}}{(V^* h^*/\nu^*)} \frac{du}{dy}; \end{aligned} \tag{2.4}$$

$$\begin{aligned} \tau_R^* &= \tau_R^*(y^*; \dots) = -\rho^* \overline{(u^{*\prime} v^{*\prime})} = \rho^* \epsilon^* \left[\frac{du^*}{dy^*} - S^*\right] \\ &= \frac{\rho^* V^{*2}}{(V^* h^*/\nu^*)} \tau_R(y; \dots) = \frac{\rho^* V^{*2}}{(V^* h^*/\nu^*)} \epsilon \left[\frac{du}{dy} - S\right]. \end{aligned} \tag{2.5}$$

In the closure model for the turbulent stress of (2.5), $\epsilon^*(y^*; \dots) = \nu^* \epsilon(y; \dots)$ is the kinematic eddy viscosity; the additional closure function $S^* = S^*(\dots) = (V^*/h^*) S(\dots)$ has been introduced to take into account the non-coincidence of the positions of maximum velocity $y_m^* = y_m^*(\dots) = h^* y_m(\dots)$ and zero turbulent stress $y_n^* = y_n^*(\dots) = h^* y_n(\dots)$, which generally occurs for combinations of V^* and P^* that yield a velocity extremum within the flow field (see figure 1), i.e. in terms of non-dimensional variables (for suitable combinations of V^* and P^*),

$$\frac{du}{dy} \to 0 \Rightarrow u \to u_m \quad \text{as} \quad y \to y_m;$$

$$\frac{du}{dy} \to S = \left(\frac{du}{dy}\right)_n \neq 0 \Rightarrow \tau_R \to 0 \quad \text{as} \quad y \to y_n \neq y_m.$$

Although this non-coincidence is well documented for duct flows with asymmetric boundary conditions (cf., e.g., Hanjalić & Launder 1972; Rehme 1974), hitherto, it has not been taken into account in the analysis of Couette–Poiseuille flows. It should be emphasized that, at this point, the (non-dimensional) function S is unknown, and must be determined in the course of the solution of the present problem.

The total shear stress is the sum of the laminar and turbulent contributions, i.e.

$$\tau^* = \tau^*(y^*; \ldots) = (\tau_N^* + \tau_R^*)$$
$$= \frac{\rho^* V^{*2}}{(V^*h^*/\nu^*)}\tau(y; \ldots) = \frac{\rho^* V^{*2}}{(V^*h^*/\nu^*)}\left(\frac{du}{dy}+\epsilon\left[\frac{du}{dy}-S\right]\right). \tag{2.6}$$

In the analysis that follows, it is taken that the Reynolds number $R \equiv (V^*h^*/\nu^*) \rightarrow \infty$.

In the domain $-\infty < x < \infty$, $0 \leqslant y \leqslant 1$, the non-dimensional boundary-value problem for fully-developed turbulent Couette–Poiseuille flow, in terms of the quantities introduced, is given by

$$-P = \frac{1}{R}\frac{d\tau}{dy} = \frac{1}{R}\frac{d}{dy}\left(\frac{du}{dy}+\epsilon\left[\frac{du}{dy}-S\right]\right) = \text{const.}; \tag{2.7}$$

$$u \rightarrow 0, \quad \epsilon \rightarrow 0 \quad \text{as} \quad y \rightarrow 0, \tag{2.8a}$$

$$u \rightarrow 1, \quad \epsilon \rightarrow 0 \quad \text{as} \quad y \rightarrow 1. \tag{2.8b}$$

Integration of (2.7) over the domain of y, subject to the boundary conditions of (2.8), yields

$$-P = \frac{1}{R}\left[\left(\frac{du}{dy}\right)_{(1)} - \left(\frac{du}{dy}\right)_{(0)}\right], \tag{2.9a}$$

for
$$\frac{du}{dy} \rightarrow \left(\frac{du}{dy}\right)_{(0)} \quad \text{as} \quad y \rightarrow 0, \quad \frac{du}{dy} \rightarrow \left(\frac{du}{dy}\right)_{(1)} \quad \text{as} \quad y \rightarrow 1. \tag{2.9b}$$

Further, from the above, it is seen that (2.7) has a first integral, which may be expressed as

$$\frac{1}{R}\left(\frac{du}{dy}+\epsilon\left[\frac{du}{dy}-S\right]\right)$$
$$= \frac{1}{R}\left(\frac{du}{dy}\right)_{(0)} - Py = \tau_{(0)}(1-\beta_{(0)}y) \tag{2.10a}$$
$$= \frac{1}{R}\left(\frac{du}{dy}\right)_{(1)} + P(1-y) = \tau_{(1)}(1-\beta_{(1)}(1-y)), \tag{2.10b}$$

where
$$\tau_{(0)} = \frac{1}{R}\left(\frac{du}{dy}\right)_{(0)}, \quad \tau_{(1)} = \frac{1}{R}\left(\frac{du}{dy}\right)_{(1)}; \tag{2.11a}$$

$$\beta_{(0)} = \frac{P}{\tau_{(0)}} = \left(1-\frac{\tau_{(1)}}{\tau_{(0)}}\right), \quad \beta_{(1)} = -\frac{P}{\tau_{(1)}} = \left(1-\frac{\tau_{(0)}}{\tau_{(1)}}\right). \tag{2.11b}$$

Further, it is noted from (2.11b) that

$$\beta_{(0)}\beta_{(1)} - (\beta_{(0)}+\beta_{(1)}) = 0 \tag{2.12a}$$

$$\Rightarrow \beta_{(0)} = -\frac{\beta_{(1)}}{(1-\beta_{(1)})} = \frac{\beta_{(1)}}{(\beta_{(1)}-1)},$$

$$\beta_{(1)} = -\frac{\beta_{(0)}}{(1-\beta_{(0)})} = \frac{\beta_{(0)}}{(\beta_{(0)}-1)}. \tag{2.12b}$$

In general, upon the specification of $\epsilon(y; R, P)$ (see § 2.2), what is sought are the solutions of the boundary-value problem (2.10a) or (2.10b) and (2.8) for $u(y; R, P)$ and

$\tau_{(0)}(R,P)$ and $\tau_{(1)}(Re,P)$. Here, the pressure gradient parameters $\beta_{(0)}(R,P)$ and $\beta_{(1)}(R,P)$ are considered to be given fixed quantities, and the asymptotic solutions to the above-mentioned boundary-value problem are sought in the limit of $R \to \infty$.† Specifically, (i) for $\tau_{(0)}(R,P) = \tau_{(0)}(R,\beta_{(0)}) \equiv u^2_{\tau(0)}(R,\beta_{(0)}) > 0$, such solutions are sought in the limit of $R \to \infty$ (with $u_{\tau(0)}(R,\beta_{(0)}) \to 0, \xi_{(0)}(R,\beta_{(0)}) = R\,u_{\tau(0)}(R,\beta_{(0)}) \to \infty$), $0 \leqslant \beta_{(0)} < 2$; while (ii) for $\tau_{(1)}(R,P) = \tau_{(1)}(R,\beta_{(1)}) \equiv u^2_{\tau(1)}(R,\beta_{(1)}) > 0$, these solutions are sought in the limit of $R \to \infty$ (with $u_{\tau(1)}(R,\beta_{(1)}) \to 0$, $\xi_{(1)}(R,\beta_{(1)}) = R\,u_{\tau(1)}(R,\beta_{(1)}) \to \infty$), $0 \leqslant \beta_{(1)} < 2$.‡ From (2.11), it is seen that $\beta_{(0)}, \beta_{(1)} = 0$ corresponds to plane Couette flow, with $P = 0$, $\tau_{(0)} = \tau_{(1)}$. Further, from (2.11), it follows that $\beta_{(0)} \to 1$, $|\beta_{(1)}| \to \infty$ corresponds to zero stress at the moving wall, i.e. $\tau_{(1)} = 0$; while $\beta_{(1)} \to 1$, $|\beta_{(0)}| \to \infty$ corresponds to zero stress at the stationary wall, i.e. $\tau_{(0)} = 0$. It is found that the case of $\beta_{(0)}, \beta_{(1)} \to 2$, with proper rescaling of velocity, corresponds to plane Poiseuille flow, where $\tau_{(1)} = -\tau_{(0)}$ (cf. Bush & Fendell 1972). The condition for a velocity extremum to exist in the flow field, i.e. $(du/dy) = 0$, such that $u = u_m$ at $y = y_m$, is seen to be $1 < \beta_{(0)} < 2$ or $1 < \beta_{(1)} < 2$.

Based upon the preceding considerations, the following boundary-value problem is stated:

$$\frac{1}{R\,u_\tau^2(R,\beta)}\left(\frac{du}{dy}(y;\,R,\beta)+\epsilon(y;\,R,\beta)\left[\frac{du}{dy}(y;\,R,\beta)-S(R,\beta)\right]\right) = (1-\beta y);$$

$$u(0;\,R,\beta) = 0, \quad u(1;\,R,\beta) = 1, \tag{2.13}$$

and $\epsilon(0;\,R,\beta) = \epsilon(1;\,R,\beta) = 0$, where $\beta = \beta_{(0)}$, $u_\tau(R,\beta) = u_{\tau(0)}(R,\beta_{(0)})$, and $\epsilon(y;\,R,\beta) = \epsilon_{(0)}(y;\,R,\,\beta_{(0)})$ for $0 \leqslant \beta < 2$. By appropriate translation and reflexion of the co-ordinate system, etc., this domain of β encompasses the entire family of Couette–Poiseuille flows. It should be noted that, for this analysis, β is considered to be a known constant of order unity, even though this parameter depends on the applied pressure gradient P and the friction velocity u_τ, through $\beta = P/u_\tau^2$. Here, the proposed method of solution is to determine $u_\tau = u_\tau(R,\beta)$ and, in turn, to determine $P = P(R,\beta) = \beta u_\tau^2(R,\beta)$. For the case of Couette flow, $\beta = 0$; for the case of Poiseuille flow, $\beta \to 2$; while, for the case of zero net flow, as previously noted, it is determined (cf. § 4.3) that $\beta = \beta(R) = 1+\alpha(R)$, with $\alpha(R) \to 0$ as $R \to \infty$. Thus, for these cases, with β specified, it is possible to determine $P = P(R)$ (cf. (4.15) for the zero net flow case). Mathematically, this approach (of obtaining solutions of the problem for $R \to \infty$ and for $\beta = O(1)$) is the only practical one available. That it is a realistic approach is justified by the results obtained, especially those for the case of zero net flow.

2.2. *The eddy-viscosity model*

In order to solve the foregoing boundary-value problem, the kinematic eddy viscosity ϵ must be specified. Since the objective of the present work is to demonstrate that Couette–Poiseuille flows can be determined with good accuracy through the use of simple closure models, the following zero-equation model, with $\epsilon = \epsilon(y;\,R,\beta)$ specified,

† In § 4.3, the case of $\beta_{(0)} \equiv \beta = \beta(R) \equiv 1+\alpha(R)$, with $\alpha(R) \to 0$ as $R \to \infty$, is studied.

‡ Consistent with these formulations, the eddy viscosity should be expressed as

$$\epsilon(y;\,R,\,P) = \epsilon_{(0)}(y;\,R,\,\beta_{(0)}) = \epsilon_{(1)}(y;\,R,\,\beta_{(1)}).$$

is adopted here (cf. Bush & Fendell 1972):

$$\epsilon(y;\,R,\beta) = \xi(R,\beta)\,K(y;\,R,\beta)$$
$$= \xi(R,\beta)\,[M(y;\,\beta)\,N_{(0)}(\eta)\,N_{(1)}(\zeta)], \tag{2.14}$$

where $$\xi(R,\beta) = \kappa R\, u_\tau(R,\beta) \rightarrow \infty \quad \text{as} \quad R \rightarrow \infty, \quad 0 \leqslant \beta < 2; \tag{2.15a}$$

$$\eta = \xi y, \quad \zeta = \xi(1-y). \tag{2.15b}$$

Thus, for $\xi \rightarrow \infty$ with y fixed, $\eta = \xi y \rightarrow \infty$ and $\zeta = \xi(1-y) \rightarrow \infty$. Further, for $\xi \rightarrow \infty$ with η fixed, $y = (\eta/\xi) \rightarrow 0$ and $\zeta = (\xi-\eta) \rightarrow \infty$; while, for $\xi \rightarrow \infty$ with ζ fixed, $y = 1-(\zeta/\xi) \rightarrow 1$ and $\eta = (\xi-\zeta) \rightarrow \infty$. For the development presented, the functions $M(y;\beta)$ and $N_{(0)}(\eta), N_{(1)}(\zeta)$, respectively, are taken to have the following representations:

$$M(y;\,\beta) = y(1-y)\,Q(y;\,\beta)$$
$$= \frac{y(1-y)}{[1+(2-\beta)\,y(1-y)+D(y;\,\beta)]}; \tag{2.16}$$

$$N_{(i)}(t) = [1-\exp\left(-t/A_{(i)}\right)]^2, \quad i = 0, 1. \tag{2.17}$$

In these equations, the following constants have been introduced: the von Kármán constant $\kappa \doteqdot 0{\cdot}41$, and the damping constants $A_{(0)}, A_{(1)} \doteqdot 6{\cdot}9$. The form of $Q(y;\,\beta)$ in (2.16) is chosen to give a continuous behaviour of the results between plane Couette flow ($\beta = 0$) and the limiting plane Poiseuille flow ($\beta \rightarrow 2$), i.e. for $0 \leqslant \beta < 2$. In (2.16), $D(y;\beta)$ has the following behaviours:

$$D(y;\,\beta) \propto \{y(1-y)\}^2 \rightarrow 0 \quad \text{as} \quad y \rightarrow 0, 1, \quad \beta \text{ fixed};$$
$$D(y;\,\beta) \propto \beta \rightarrow 0 \quad \text{as} \quad \beta \rightarrow 0, \quad y \text{ fixed}.$$

In § 3.5, relations are derived which $D(y;\,\beta)$ must satisfy for $1 < \beta < 2$. In (2.16), the linear behaviour of $M(y;\,\beta)$ as $y \rightarrow 0$, 1 is consistent with the logarithmic velocity distribution required by Prandtl's law of the wall and von Kármán's velocity-defect law (cf., e.g., White 1974). In (2.17), the form of $N_{(i)}(t)$ is the eddy-viscosity version of the Van Driest mixing-length 'exponential' damping function (cf., e.g., White 1974; Reynolds 1976).

2.3. *The basic formulation*

For the analysis, based on the given eddy-viscosity model, consider the following restatement of the boundary-value problem. The governing differential equation can now be written as

$$y(1-y)\,Q(y;\,\beta)\,N_{(0)}(\xi y)\,N_{(1)}(\xi(1-y))\left[\frac{du}{dy}(y;\,\xi,\beta)-S(\xi,\beta)\right]$$
$$+\frac{1}{\xi}\frac{du}{dy}(y;\,\xi,\beta)-\tilde{u}_\tau(\xi,\beta)\,(1-\beta y) = 0, \tag{2.18}$$

with $\tilde{R} = \kappa^2 R$, $\tilde{u}_\tau = u_\tau/\kappa$, such that $\xi = \tilde{R}\tilde{u}_\tau$. The relevant boundary conditions can now be written as

$$u(0;\,\xi,\beta) = 0, \quad u(1;\,\xi,\beta) = 1. \tag{2.19}$$

Since (2.18) is a first-order equation with two boundary conditions (i.e. those of (2.19)), (2.18) and (2.19) constitute an eigenvalue problem, where $\tilde{u}_\tau(\xi,\beta)$ is the eigenvalue to be determined in the limit of $\xi \to \infty$ for $0 \leqslant \beta < 2$. Additionally, the solution of this eigenvalue problem leads to the determination of $S(\xi, \beta)$ as $\xi \to \infty$ for $1 < \beta < 2$.

3. Asymptotic solutions

In this section, the solutions to the singular perturbation eigenvalue problem of (2.18) and (2.19) are sought. These solutions are obtained through the use of limit process expansion techniques (cf., e.g., Cole 1968; Bush & Fendell 1972, 1974), where the velocity field in the core region of the duct is matched to the velocity fields in the regions near the two walls. The eigenvalue $\tilde{u}_\tau(\xi, \beta)$, which is determined from the matching requirements, leads to the friction law $C_f = C_f(R, \beta)$, where the friction coefficient is defined by $C_f = 2u_\tau^2 = 2(\kappa\tilde{u}_\tau)^2$.

3.1. *Core region expansions*

When the core region of the duct is taken to be a turbulent defect layer (cf. Bush & Fendell, 1972, 1974), the appropriate spatial variable is y, and the complementary velocity variable is

$$F(y; \xi, \beta) = \frac{u(y; \xi, \beta) - U(\beta)}{\tilde{u}_\tau(\xi, \beta)}, \tag{3.1}$$

where, subject to verification, it is taken that $\tilde{u}_\tau(\xi, \beta) \to 0$ as $\xi \to \infty$, β fixed. Thus, for this limit, $u(y; \xi, \beta) \to U(\beta)$, a constant (to be determined). Since $\eta = \xi y \to \infty$ and $\zeta = \xi(1-y) \to \infty$, such that $N_{(0)}(\eta), N_{(1)}(\zeta) \to [1 - O(\exp(-\xi))]$, as $\xi \to \infty$, with y fixed, (2.18), in terms of the core defect layer variables, is given by

$$y(1-y)\,Q(y; \beta)\left[\frac{dF}{dy}(y; \xi, \beta) - H(\xi, \beta)\right] = (1-\beta y) - \frac{1}{\xi}\frac{dF}{dy}(y; \xi, \beta) + O(\exp(-\xi)), \tag{3.2}$$

where $H(\xi, \beta) = S(\xi, \beta)/\tilde{u}_\tau(\xi, \beta)$.

Consider that the function $F(y; \xi, \beta)$ and the parameter $H(\xi, \beta)$ are taken to have the following asymptotic expansions:

$$F(y; \xi, \beta) \simeq F_0(y; \beta) + \mu_{01}(\xi)\,F_{01}(y; \beta) + \frac{1}{\xi}F_1(y; \beta) + \ldots, \tag{3.3a}$$

$$H(\xi, \beta) \simeq H_0(\beta) + \mu_{01}(\xi)\,H_{01}(\beta) + \frac{1}{\xi}H_1(\beta) + \ldots, \tag{3.3b}$$

with $\mu_{01}(\xi)$, a parameter (to be determined), which satisfies $\mu_{01}(\xi) \to 0$, $\xi\mu_{01}(\xi) \to \infty$ as $\xi \to \infty$.

Substitution of (3.3) into (3.2) and collection of terms of equal order yields

$$y(1-y)\,Q(y; \beta)\left[\frac{dF_0}{dy}(y; \beta) - H_0(\beta)\right] = (1-\beta y); \tag{3.4a}$$

$$y(1-y)\,Q(y; \beta)\left[\frac{dF_{01}}{dy}(y; \beta) - H_{01}(\beta)\right] = 0; \quad \ldots. \tag{3.4b}$$

The solutions of (3.4*a*) and (3.4*b*) are

$$F_0(y; \beta) = H_0(\beta)\,y + C_0(\beta) + \{\log y - (1-\beta)\log(1-y)\} + \tfrac{1}{2}(2-\beta)\,y(2-\beta y) + \Theta_0(y; \beta), \tag{3.5a}$$

$$F_{01}(y; \beta) = H_{01}(\beta)\,y + C_{01}(\beta), \tag{3.5b}$$

where $C_0(\beta)$, $C_{01}(\beta)$ are integration constants (to be determined), and

$$\Theta_0(y;\beta) = \int_0^y \left[\frac{(1-\beta t)}{t(1-t)}\right] D(t;\beta)\,dt, \quad \text{with} \quad \Theta_0(1;\beta) \equiv \Psi_0(\beta). \tag{3.5c}$$

These solutions are based on $Q(y;\beta)$ as defined by (2.16). Thus the asymptotic representation for the core region velocity is

$$\begin{aligned} u(y;\xi,\beta) &= u_{(c)}(y;\xi,\beta) \\ &= U(\beta) + \tilde{u}_\tau(\xi,\beta)\left(F_0(y;\beta) + \mu_{01}(\xi)F_{01}(y;\beta) + O\left(\frac{1}{\xi}\right)\right). \end{aligned} \tag{3.6}$$

where the functions F_0, F_{01} are given in (3.5)

For the purpose of matching the core region velocity solution to those for the wall regions, the behaviours of $u_{(c)}(y;\xi,\beta)$ as $y \to 0, 1$ are required. For $y \to 0$, (3.6) yields

$$\begin{aligned} u_{(c)}(y;\xi,\beta) \sim U(\beta) + \tilde{u}_\tau(\xi,\beta)\Bigg(\Bigg[-\log\left(\frac{1}{y}\right) + C_0(\beta) + \{H_0(\beta) + (3-2\beta)\}y + \ldots\Bigg] \\ + \mu_{01}(\xi)\,[C_{01}(\beta) + H_{01}(\beta)y] + \ldots\Bigg). \end{aligned} \tag{3.7a}$$

For $z = (1-y) \to 0$, (3.6) yields

$$\begin{aligned} u_{(c)}(y;\xi,\beta) \sim U(\beta) + \tilde{u}_\tau(\xi,\beta)\Bigg(\Bigg[(1-\beta)\log\left(\frac{1}{z}\right) + \{C_0(\beta) + H_0(\beta) + \tfrac{1}{2}(2-\beta)^2 + \Psi_0(\beta)\} \\ - \{H_0(\beta) + 1 + (1-\beta)(2-\beta)\}z + \ldots\Bigg] \\ + \mu_{01}(\xi)\,[\{C_{01}(\beta) + H_{01}(\beta)\} - H_{01}(\beta)z] + \ldots\Bigg). \end{aligned} \tag{3.7b}$$

When the core region velocity solutions are matched to the wall regions velocity solutions, it is found that the intermediate terms of $O(\mu_{01}(\xi)), \ldots$ are necessary to achieve the matching for non-zero $H_0(\beta), H_{01}(\beta), H_1(\beta), \ldots$. Although these intermediate terms, which do not appear in symmetric duct flows (cf. Bush & Fendell 1972), are necessary for matching at this stage, arguments are presented in § 3.5 that these contributions, i.e. $H_{01}(\beta), \ldots$, are zero.

3.2. *Stationary wall region expansions*

For the stationary wall region of the duct, the appropriate spatial and velocity variables are taken to be

$$\eta = \xi y; \quad f(\eta;\xi,\beta) = \frac{u(y;\xi,\beta)}{\tilde{u}_\tau(\xi,\beta)}. \tag{3.8), (3.9}$$

Since $y = (\eta/\xi) \to 0$ and $\zeta = (\xi-\eta) \to \infty$, such that $Q(\eta/\xi;\beta) \to [1-(2-\beta)(\eta/\xi) + O(1/\xi^2)]$ and $N_{(1)}(\xi-\eta) \to [1-O(\exp(-\xi))]$, as $\xi \to \infty$, with η fixed, (2.18) and (2.19), in terms of these variables, with $N_{(0)}(\eta) \equiv N(\eta)$, are

$$\begin{gathered} [1+\eta N(\eta)]\frac{df}{d\eta}(\eta;\xi,\beta) = 1 + \frac{\eta}{\xi}\left\{N(\eta)\left[(3-\beta)\eta\frac{df}{d\eta}(\eta;\xi,\beta) + H(\xi,\beta)\right] - \beta\right\} + O\left(\frac{1}{\xi^2}\right); \\ f(0;\xi,\beta) = 0. \end{gathered} \tag{3.10}$$

Consider that the function $f(\eta;\xi,\beta)$ is taken to have the following asymptotic expansion:

$$f(\eta;\xi,\beta) \cong f_0(\eta;\beta)+\frac{1}{\xi}f_1(\eta;\beta)+\frac{\mu_{01}(\xi)}{\xi}f_{12}(\eta;\beta)+\frac{1}{\xi^2}f_2(\eta;\beta)+\dots \tag{3.11}$$

Substitution of (3.11) into (3.10) and collection of terms of equal order yields

$$[1+\eta N(\eta)]\frac{df_0}{d\eta}(\eta;\beta) = 1, \quad f_0(0;\beta)=0; \tag{3.12a}$$

$$[1+\eta N(\eta)]\frac{df_1}{d\eta}(\eta;\beta) = \eta\left\{N(\eta)\left[(3-\beta)\eta\frac{df_0}{d\eta}(\eta;\beta)+H_0(\beta)\right]-\beta\right\}, \quad f_1(0;\beta)=0; \tag{3.12b}$$

$$[1+\eta N(\eta)]\frac{df_{12}}{d\eta}(\eta;\beta) = \eta N(\eta)H_{01}(\beta), \quad f_{12}(0;\beta)=0. \tag{3.12c}$$

The solutions of (3.12a,b,c) are

$$f_0(\eta;\beta) = \log(1+\eta)+I_0(\eta), \tag{3.13a}$$

$$f_1(\eta;\beta) = \{H_0(\beta)+(3-2\beta)\}\eta-\{H_0(\beta)+3(2-\beta)\}\log(1+\eta)$$
$$+(3-\beta)\frac{\eta}{(1+\eta)}-\{H_0(\beta)I_0(\eta)+\beta I_{01}(\eta)+(3-\beta)I_1(\eta)\}, \tag{3.13b}$$

$$f_{12}(\eta;\beta) = H_{01}(\beta)\{\eta-\log(1+\eta)-I_0(\eta)\}, \tag{3.13c}$$

where $$I_0(\eta)=\int_0^\eta\left[\frac{t}{(1+t)}\right]\left[\frac{1-N(t)}{1+tN(t)}\right]dt, \quad \text{with} \quad I_0(\infty)\equiv J_0, \tag{3.14a}$$

$$I_{01}(\eta)=\int_0^\eta\left[\frac{t^2}{(1+t)}\right]\left[\frac{1-N(t)}{1+tN(t)}\right]dt, \quad \text{with} \quad I_{01}(\infty)\equiv J_{01}, \tag{3.14b}$$

$$I_1(\eta)=\int_0^\eta\left[\frac{t^2}{(1+t)^2}\right]\left[\frac{1-N(t)}{1+tN(t)}\right]\left[\frac{1-t^2N(t)}{1+tN(t)}\right]dt, \quad \text{with} \quad I_1(\infty)\equiv J_1. \tag{3.14c}$$

Thus, the asymptotic representation for the stationary wall region velocity is

$$u(y;\xi,\beta) = u_{(0)}(\eta;\xi,\beta)$$
$$=\tilde{u}_\tau(\xi,\beta)\left(f_0(\eta;\beta)+\frac{1}{\xi}f_1(\eta;\beta)+\frac{\mu_{01}(\xi)}{\xi}f_{12}(\eta;\beta)+O\left(\frac{1}{\xi^2}\right)\right), \tag{3.15}$$

where the functions f_0, f_1, f_{12} are given in (3.13).

The behaviour of $u_{(0)}(\eta;\xi,\beta)$ as $\eta\to\infty$ is required in order to match this stationary wall region velocity solution to that for the core region, i.e., $u_{(c)}(y;\xi,\beta)$ as $y\to 0$. For $\eta\to\infty$, (3.15) yields

$$u_{(0)}(\eta;\xi,\beta) \sim \tilde{u}_\tau(\xi,\beta)\Big([\log\eta+J_0+\dots]$$
$$+\frac{1}{\xi}[\{H_0(\beta)+(3-2\beta)\}\eta-\{H_0(\beta)+3(2-\beta)\}\log\eta+\dots]$$
$$+\frac{\mu_{01}(\xi)}{\xi}[H_{01}(\beta)\eta+\dots]+\dots\Big). \tag{3.16†}$$

† In the solution of (3.12a), the integral (3.14a) is a consequence of $\epsilon(y;\xi,\beta)\to y$ as $y\to 0$, modified by $N_{(0)}(\eta)$, as in (2.14). Since this linearity is required (cf. (2.16), *et seq.*), and since it is taken that the damping function depends only on η and not on β, $I_0(\eta)$ can be regarded as a

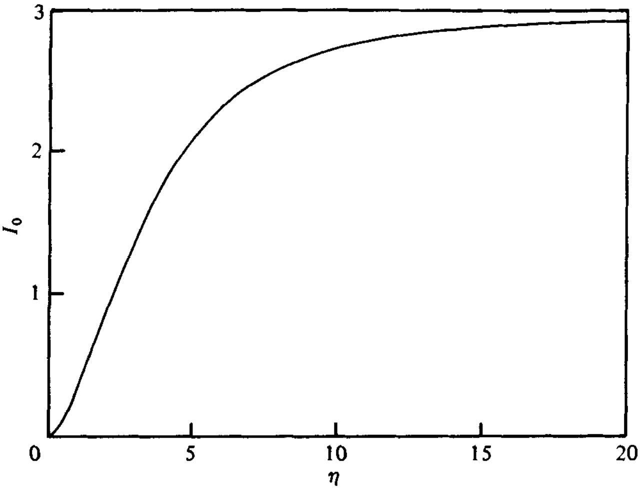

FIGURE 2. Numerical evaluation of (3.14a).

A comparison of (3.16) and (3.7a) shows that the stationary wall region and core region velocity solutions match *if*

$$U(\beta)+\tilde{u}_\tau(\xi,\beta)\,[C_0(\beta)+\mu_{01}(\xi)\,C_{01}(\beta)+\ldots]$$
$$=\tilde{u}_\tau(\xi,\beta)\,[\log\xi+J_0-\mu_{01}(\xi)\,\{H_0(\beta)+3(2-\beta)\}+\ldots], \qquad (3.17a)$$

with
$$\mu_{01}(\xi)=\log\xi. \qquad (3.17b)$$

For further details concerning the matching process as applied to turbulent channel flow, the reader is referred to the papers by Bush & Fendell (1972, 1974). This matching condition (3.17) will be used, together with a complementary matching condition for the moving wall region and core region velocity solutions, to determine the eigenvalue $\tilde{u}_\tau(\xi,\beta)$.

3.3. *Moving wall region expansions*

For the moving wall region of the duct, the appropriate spatial and velocity variables are taken to be

$$\zeta=\xi(1-y);\quad g(\zeta;\xi,\beta)=\frac{1-u(y;\xi,\beta)}{\tilde{u}_\tau(\xi,\beta)}, \qquad (3.18),\ (3.19)$$

---

universal function for all duct flows and its limit $I_0(\infty)=J_0$ as a universal constant similar to von Kármán's constant κ. Figure 2 shows the numerical determination of $I_0(\eta)$ for

$$N(t)=[1-\exp(-t/A_{(0)})]^2,$$

with $A_{(0)}\doteqdot 6{\cdot}9$. As can be seen, $I_0(\eta)$ approaches a fixed limit as $\eta\to\infty$, namely, $J_0\doteqdot 2{\cdot}95$. In terms of the conventional wall region variables $y^+=y^*u_\tau^*/\nu^*=\eta/\kappa$ and $u^+=u^*/u_\tau^*=f/\kappa$, (3.16) yields, to leading order of approximation,

$$u^+=\frac{1}{\kappa}\log y^+ + b,\quad \text{where}\quad b=\frac{1}{\kappa}\left\{J_0-\log\left(\frac{1}{\kappa}\right)\right\}.$$

For $\kappa\doteqdot 0{\cdot}41$ and $J_0\doteqdot 2{\cdot}95$, it is found that $b\doteqdot 5{\cdot}0$, which is the accepted value (cf., e.g., White 1974; Hussain & Reynolds 1975).

As $\eta\to 0$, $I_0(\eta)\sim\frac{1}{2}\eta^2[1-\frac{2}{3}\eta+O(\eta^2)]\to 0$. Thus, (3.13$a$) yields $f_0(\eta;\beta)\sim\eta[1+O(\eta^3)]\to 0$ as $\eta\to 0$, which is the characteristic linear velocity distribution of the viscous sublayer very near the wall (cf. Bush & Fendell 1974).

Since $y = 1-(\zeta/\xi) \to 1$ and $\eta = (\xi-\zeta) \to \infty$, as $\xi \to \infty$, with ζ fixed, in terms of these variables, with $N_{(1)}(\zeta) \equiv N(\zeta)$, (2.18) and (2.19) can be written as

$$[1+\zeta N(\zeta)]\frac{dg}{d\zeta}(\zeta;\xi,\beta) = (1-\beta)+\frac{\zeta}{\xi}\left\{N(\zeta)\left[(3-\beta)\zeta\frac{dg}{d\zeta}(\zeta;\ \xi,\beta)+H(\xi,\beta)\right]+\beta\right\}+O\left(\frac{1}{\xi^2}\right);$$
$$g(0;\xi,\beta) = 0. \tag{3.20}$$

The function $g(\zeta;\xi,\beta)$ is taken to have the following asymptotic expansion:

$$g(\zeta;\ \xi,\beta) \cong g_0(\zeta;\beta)+\frac{1}{\xi}g_1(\zeta;\beta)+\frac{\mu_{01}(\xi)}{\xi}g_{12}(\zeta;\beta)+\frac{1}{\xi^2}g_2(\zeta;\beta)+\dots . \tag{3.21}$$

Substitution of (3.21) into (3.20) and collection of terms of equal order yields

$$[1+\zeta N(\zeta)]\frac{dg_0}{d\zeta}(\zeta;\beta) = (1-\beta), \quad g_0(0;\beta) = 0; \tag{3.22a}$$

$$[1+\zeta N(\zeta)]\frac{dg_1}{d\xi}(\zeta;\beta) = \zeta\left\{N(\zeta)\left[(3-\beta)\zeta\frac{dg_0}{d\zeta}(\zeta;\beta)+H_0(\beta)\right]+\beta\right\}, \quad g_1(0;\beta); \tag{3.22b}$$

$$[1+\zeta N(\zeta)]\frac{dg_{12}}{d\zeta}(\zeta;\beta) = \zeta N(\zeta)H_{01}(\beta), \quad g_{12}(0;\beta) = 0. \tag{3.22c}$$

The solutions of ($3.22a, b, c$) are

$$g_0(\zeta;\beta) = (1-\beta)[\log(1+\zeta)+I_0(\zeta)\}, \tag{3.23a}$$

$$g_1(\zeta;\beta) = \{H_0(\beta)+1+(1-\beta)(2-\beta)\}\zeta-\{H_0(\beta)+(3-2\beta)(2-\beta)\}\log(1+\zeta)$$
$$+(1-\beta)(3-\beta)\frac{\zeta}{(1+\zeta)}-\{H_0(\beta)I_0(\zeta)-\beta I_{01}(\zeta)+(1-\beta)(3-\beta)I_1(\zeta)\}, \tag{3.23b}$$

$$g_{12}(\zeta;\beta) = H_{01}(\beta)\{\zeta-\log(1+\zeta)-I_0(\zeta)\}. \tag{3.23c}$$

Thus, the asymptotic representation for the moving wall region velocity is

$$u(y;\xi,\beta) = u_{(1)}(\zeta;\ \xi,\beta)$$
$$= 1-\tilde{u}_\tau(\xi,\beta)\left(g_0(\zeta;\beta)+\frac{1}{\xi}g_1(\zeta;\beta)+\frac{\mu_{01}(\xi)}{\xi}g_{12}(\zeta;\beta)+O\left(\frac{1}{\xi^2}\right)\right), \tag{3.24}$$

where the functions g_0, g_1, g_{12} are given in (3.23).

For $\zeta \to \infty$, (3.23) and (3.24) yield

$$u_{(1)}(\zeta;\xi,\beta) \sim 1-\tilde{u}_\tau(\xi,\beta)\Big([(1-\beta)\log\zeta+(1-\beta)J_0+\dots]$$
$$+\frac{1}{\xi}[\{H_0(\beta)+1+(1-\beta)(2-\beta)\}\zeta$$
$$-\{H_0(\beta)+(3-2\beta)(2-\beta)\}\log\zeta+\dots]$$
$$+\frac{\mu_{01}(\xi)}{\xi}[H_{01}(\beta)\zeta+\dots]+\dots\Big). \tag{3.25}$$

A comparison of (3.25) with ($3.7b$) shows that the moving wall region and core region velocity solutions match *if*

$$U(\beta)+\tilde{u}_\tau(\xi,\beta)\,[\{C_0(\beta)+H_0(\beta)+\tfrac{1}{2}(2-\beta)^2+\Psi_0(\beta)\}+\mu_{01}(\xi)\,\{C_{01}(\beta)+H_{01}(\beta)\}+\dots]$$
$$=1-\tilde{u}_\tau(\xi,\beta)\,[(1-\beta)\log\xi+(1-\beta)\,J_0-\mu_{01}(\xi)\,\{H_0(\beta)+(3-2\beta)\,(2-\beta)\}+\dots]. \quad (3.26)$$

This equation, (3.26), is the other matching condition which, in combination with (3.17), yields $\tilde{u}_\tau(\xi,\beta)$.

3.4. *The friction law*

For what follows, it is convenient to re-express (3.17) and (3.26), respectively, as

$$U(\beta)-\tilde{u}_\tau(\xi,\beta)\,[\log\xi-\{C_0(\beta)-J_0\}$$
$$-\frac{\log\xi}{\xi}\{C_{01}(\beta)+H_0(\beta)+3(2-\beta)\}+\dots]=0; \quad (3.27a)$$

$$U(\beta)+\tilde{u}_\tau(\xi,\beta)\,[(1-\beta)\log\xi+\{C_0(\beta)+H_0(\beta)+\tfrac{1}{2}(2-\beta)^2+\Psi_0(\beta)+(1-\beta)\,J_0\}$$
$$+\frac{\log\xi}{\xi}\{C_{01}(\beta)+H_{01}(\beta)-H_0(\beta)-(3-2\beta)\,(2-\beta)\}+\dots]=1. \quad (3.27b)$$

The combination $[(1-\beta)\times(3.27a)+(3.27b)]$ yields

$$\hat{U}(\beta)+\hat{u}_\tau(\hat{\xi},\beta)\,[\{C_0(\beta)+\hat{H}_0(\beta)+\tfrac{1}{2}(2-\beta)+\hat{\Psi}_0(\beta)\}$$
$$+\frac{\log\hat{\xi}}{\hat{\xi}}\{C_{01}(\beta)+\hat{H}_{01}(\beta)-\beta(\hat{H}_0(\beta)+1)\}+\dots]=1, \quad (3.28)$$

where
$$\hat{\xi}=\xi=\tilde{u}_\tau\tilde{R}=\Big[(2-\beta)\,\tilde{u}_\tau\Big]\left[\frac{\tilde{R}}{(2-\beta)}\right]=\hat{u}_\tau\hat{R}; \quad (3.29a)$$
$$\hat{U}=(2-\beta)\,U; \quad (3.29b)$$
$$\hat{H}_0=\frac{H_0}{(2-\beta)},\quad \hat{\Psi}_0=\frac{\Psi_0}{(2-\beta)},\quad \hat{H}_{01}=\frac{H_{01}}{(2-\beta)},\quad \dots. \quad (3.29c)$$

From (3.28), it follows that

$$\hat{U}(\beta)=1\Rightarrow U(\beta)=\frac{1}{(2-\beta)}; \quad (3.30a)$$

$$C_0(\beta)=-\{\hat{H}_0(\beta)+\tfrac{1}{2}(2-\beta)+\hat{\Psi}_0(\beta)\},$$
$$C_{01}(\beta)=-\{\hat{H}_{01}(\beta)-\beta(\hat{H}_0(\beta)+1)\},\quad\dots. \quad (3.30b)$$

The combination $[(3.27b)-(3.27a)]$ yields

$$\hat{u}_\tau(\hat{\xi},\beta)\left[\log\hat{\xi}+\hat{E}_0(\beta)+\hat{E}_{01}(\beta)\frac{\log\hat{\xi}}{\hat{\xi}}+\dots\right]=1, \quad (3.31a)$$

and/or

$$\hat{\xi}(\hat{R},\beta)\log\hat{\xi}\,(\hat{R},\beta)\left[1+\frac{\hat{E}_0(\beta)}{\log\hat{\xi}(\hat{R},\beta)}+\frac{\hat{E}_{01}(\beta)}{\hat{\xi}(\hat{R},\beta)}+\dots\right]=\hat{R}, \quad (3.31b)$$

where
$$\hat{E}_0(\beta)=J_0+\hat{H}_0(\beta)+\tfrac{1}{2}(2-\beta)+\hat{\Psi}_0(\beta), \quad (3.31c)$$
$$\hat{E}_{01}(\beta)=\hat{H}_{01}(\beta)-2\{\hat{H}_0(\beta)+(3-\beta)\},\quad\dots. \quad (3.31d)$$

The asymptotic solution of (3.31b) for $\hat{R}e\to\infty$, β fixed is determined to be

$$\hat{\xi}(\hat{R},\beta)\cong\frac{\hat{R}}{\log\hat{R}}\left[1+\frac{\log\log\hat{R}-\hat{E}_0(\beta)}{\log\hat{R}}+\dots\right]. \quad (3.32a)$$

Thus, it follows that

$$\hat{u}_\tau(\hat{R},\beta) = \frac{\hat{\xi}(\hat{R},\beta)}{\hat{R}} \cong \frac{1}{\log \hat{R}}\left[1+\frac{\log\log \hat{R} - \hat{E}_0(\beta)}{\log \hat{R}}+\ldots\right]. \tag{3.32b}$$

In turn, it is determined, based upon (3.29*a*), that

$$\xi(\tilde{R},\beta) \cong \frac{1}{(2-\beta)}\frac{\tilde{R}}{\log \tilde{R}}\left(1+\frac{\log\log \tilde{R}+\log(2-\beta)-\tilde{E}_0(\beta)}{\log \tilde{R}}+\ldots\right], \tag{3.33a}$$

$$\tilde{u}_\tau(\tilde{R},\beta) \cong \frac{1}{(2-\beta)}\frac{1}{\log \tilde{R}}\left[1+\frac{\log\log \tilde{R}+\log(2-\beta)-\tilde{E}_0(\beta)}{\log \tilde{R}}+\ldots\right], \tag{3.33b}$$

where
$$\tilde{E}_0(\beta) = \hat{E}_0(\beta) = J_0+\frac{1}{(2-\beta)}H_0(\beta)+\tfrac{1}{2}(2-\beta)+\frac{1}{(2-\beta)}\Upsilon_0(\beta). \tag{3.33c}$$

Thus, it is seen from (3.33) that $\tilde{u}_\tau(\tilde{R},\beta)\to 0$ and $\xi(\tilde{R},\beta) = \tilde{R}\,\tilde{u}_\tau(\tilde{R},\beta)\to\infty$ as $\tilde{R}\to\infty$, β fixed, as was presumed at the beginning of the analysis. These equations, (3.33*a*, *b*), have the same asymptotic forms as the ones obtained by Bush & Fendell (1972, 1974) for plane duct flow without a moving wall. By definition, the friction coefficient is

$$C_f(R,\beta) = 2u_\tau^2(R,\beta)\to 0 \quad \text{as} \quad R\to\infty, \quad \beta \text{ fixed}, \tag{3.34a}$$

where
$$u_\tau(R,\beta) = \kappa\tilde{u}_\tau(\tilde{R},\beta), \quad \text{with} \quad \tilde{R} = \kappa^2 R,$$

$$= \frac{\kappa}{(2-\beta)}\hat{u}_\tau(\hat{R},\beta), \quad \text{with} \quad \hat{R} = \frac{\kappa^2}{(2-\beta)}R. \tag{3.34b}$$

The asymptotic solutions for $\hat{u}_\tau(\hat{R},\beta)$ and $\tilde{u}_\tau(\tilde{R},\beta)$, respectively, are given in (3.32*b*) and (3.33*b*). Further, with the determination of $u_\tau(R,\beta)$, the pressure gradient function $P(R,\beta)$ follows, i.e.

$$P(R,\beta) = \beta u_\tau^2(R,\beta) = \tfrac{1}{2}\beta C_f(R,\beta)\to 0 \quad \text{as} \quad R\to\infty, \quad \beta \text{ fixed}. \tag{3.35}$$

With the evaluation of the constants (cf. (3.30)), to leading orders of approximation, the core region velocity solution (cf. (3.6)) can be re-expressed as

$$\begin{aligned} u_{(c)}(y;\,\xi,\beta) \cong \frac{1}{(2-\beta)}+\tilde{u}_\tau(\xi,\beta)\Bigg(\Bigg[&\{\log y-(1-\beta)\log(1-y)\} \\ &-H_0(\beta)\left\{\frac{1}{(2-\beta)}-y\right\}-\tfrac{1}{2}(2-\beta)\{1-y(2-\beta y)\} \\ &-\left\{\frac{1}{(2-\beta)}\Upsilon_0(\beta)-\Theta_0(y;\beta)\right\}\Bigg]+\ldots\Bigg). \end{aligned} \tag{3.36}$$

To leading orders of approximation, the stationary wall region and moving wall region velocity solutions of (3.15) and (3.24), respectively, are

$$u_{(0)}(\eta;\,\xi,\beta) \cong \tilde{u}_\tau(\xi,\beta)\,([\log(1+\eta)+I_0(\eta)]+\ldots); \tag{3.37a}$$

$$u_{(1)}(\zeta;\,\xi,\beta) \cong 1-\tilde{u}_\tau(\xi,\beta)\,([(1-\beta)\{(\log(1+\zeta)+I_0(\zeta)\}]+\ldots). \tag{3.37b}$$

Through the combination of these asymptotic representations, a composite expansion can be obtained (cf. Cole 1968), which is uniformly valid across the duct as $\xi\to\infty$, β fixed. This uniformly valid asymptotic representation for the velocity is determined to be

$$\begin{aligned} u^{(c)}(y;\,\xi,\beta) \cong \frac{(1-\beta)}{(2-\beta)}+\tilde{u}_\tau(\xi,\beta)\Bigg(\Bigg[&\{H_0(\beta)y+\tfrac{1}{2}(2-\beta)y(2-\beta y)+\Theta_0(y;\beta)\} \\ +\{\log(1+\xi y)+I_0(\xi y)\}&-(1-\beta)\{\log(1+\xi(1-y))+I_0(\xi(1-y))\} \\ &-\frac{(1-\beta)}{(2-\beta)}\{H_0(\beta)+\tfrac{1}{2}(2-\beta)^2+\Upsilon_0(\beta)\}\Bigg]+\ldots\Bigg). \end{aligned} \tag{3.38}$$

3.5. *Determination of* $H(\xi,\beta)$

From the analysis presented, the ξ-dependence of $H(\xi,\beta)$ has been determined (cf. (3.3*b*)), but the β-dependence, represented by $H_0(\beta), H_{01}(\beta), \ldots$, is, as yet, not specified. Let $y_d(\beta)$ be the co-ordinate where the core region velocity defect is zero (cf. (3.1)). Thus, by definition,

$$F(y_d(\beta);\, \xi,\beta) = 0. \qquad (3.39)$$

In the range $1 < \beta < 2$, the velocity profile has a local extremum at $y_m(\beta)$. Therefore,

$$\frac{dF}{dy}(y_m(\beta);\, \xi,\beta) = 0, \quad 1 < \beta < 2. \qquad (3.40)$$

In this range of β, $y_d(\beta) = y_m(\beta)$; otherwise, $y_d(\beta)$ would not be a single-valued function. Hence, for $1 < \beta < 2$, (3.39) and (3.40) provide sufficient information to determine $H_0(\beta), H_{01}(\beta), \ldots$. Substitution of (3.3*a*) into (3.39) yields

$$F_0(y_d(\beta);\, \beta),\ F_{01}(y_d(\beta);\, \beta), \ldots = 0,$$

or, based on (3.5) and (3.30),

$$\{\log y_d(\beta) + (\beta-1)\log(1-y_d(\beta))\} - H_0(\beta)\left\{\frac{1}{(2-\beta)} - y_d(\beta)\right\}$$
$$-\tfrac{1}{2}(2-\beta)\{1-y_d(\beta)(2-\beta y_d(\beta))\} - \left\{\frac{1}{(2-\beta)}\Psi_0(\beta) - \Theta_0(y_d(\beta);\,\beta)\right\} = 0, \qquad (3.41a)$$

$$-H_{01}(\beta)\left\{\frac{1}{(2-\beta)} - y_d(\beta)\right\} + \beta\left\{\frac{1}{(2-\beta)}H_0(\beta) + 1\right\} = 0, \ldots. \qquad (3.41b)$$

Similarly,

$$\frac{dF_0}{dy}(y_d(\beta);\,\beta),\quad \frac{dF_{01}}{dy}(y_d(\beta);\,\beta),\quad \ldots = 0, \quad \text{for} \quad 1 < \beta < 2,$$

or, based on (3.4),

$$H_0(\beta) + \frac{(1-\beta y_d(\beta))}{y_d(\beta)(1-y_d(\beta))}[1 + (2-\beta)y_d(\beta)(1-y_d(\beta)) + D(y_d(\beta);\,\beta)] = 0, \qquad (3.42a)$$

$$H_{01}(\beta) = 0, \ldots, \quad \text{for} \quad 1 < \beta < 2. \qquad (3.42b)$$

Substitution of (3.42*b*) into (3.41*b*) yields

$$H_0(\beta) = -(2-\beta) \quad \text{and/or} \quad \hat{H}_0(\beta) = -1 \quad \text{for} \quad 1 < \beta < 2. \qquad (3.43)$$

Thus, $H_0(\beta) \to 0$ as $\beta \to 2$, which is required for the limiting case of plane Poiseuille flow (i.e. y_m and y_n coincide as $\beta \to 2$, as symmetry demands).

In turn, it follows that, for $1 < \beta < 2$, (3.42*a*) and (3.41*a*), respectively, with $H_0(\beta) = -(2-\beta)$, yield

$$D(y_d(\beta);\,\beta) = \frac{\beta y_d(\beta)[1 + (2-\beta)y_d(\beta)(1-y_d(\beta))] - 1}{1-\beta y_d(\beta)}, \qquad (3.44a)$$

$$\frac{1}{(2-\beta)}\Psi_0(\beta) - \Theta_0(y_d(\beta);\,\beta) = \{\log y_d(\beta) + (\beta-1)\log(1-y_d(\beta))\}$$
$$+ 1 - \tfrac{1}{2}(2-\beta)(1+\beta y_d^2(\beta)). \qquad (3.44b)$$

Thus, with $\Psi_0(\beta)$ and $\Theta_0(y_d(\beta);\,\beta)$ specified by (3.5*c*), it is seen that (3.44*a*) and (3.44*b*) form an integral relation in $y_d(\beta)$ and $D(y_d(\beta);\,\beta)$, which any proposed closure function must satisfy. Although the asymptotic theory cannot generate the closure functions for a particular problem, it does provide constraining conditions which proposed functions must satisfy for correct modelling, as has been shown previously (cf. Bush & Fendell 1973).

Hence, (3.43) shows that $H_0(\beta) < 0$ for $1 < \beta < 2$, i.e. $y_n(\beta)$ is displaced farther from the (stationary) wall with the larger shearing force than is $y_m(\beta)$, which is true for other asymmetric duct flows (cf. Hanjalić & Launder 1972; Rehme 1974). Previous numerical Couette–Poiseuille flow analyses have assumed $H(\xi,\beta) \equiv 0$ for all β (cf., e.g., Elrod & Ng 1967; Ho & Vohr 1974; Huey & Williamson 1974).

The result, (3.43), implies that the intermediate term, $F_{01}(y;\beta)$, in the core region expansion (cf. (3.3*a*)) is zero for $1 < \beta < 2$. If this intermediate term had not been introduced initially, (3.43) would have resulted from matching requirements over the whole domain $0 \leqslant \beta < 2$, i.e.

$$F_{01}(y;\beta) \equiv 0 \quad \text{for} \quad 0 \leqslant \beta < 2,$$

$$\Rightarrow H_{01}(\beta)\,y - \frac{1}{(2-\beta)}\{H_{01}(\beta) - \beta[H_0(\beta) + (2-\beta)]\} = 0:$$

$$H_{01}(\beta) = 0; \quad H_0(\beta) = -(2-\beta) \quad \text{for} \quad 0 \leqslant \beta < 2. \tag{3.45}$$

Now, because the Couette flow ($\beta = 0$) velocity profile is antisymmetric, it might be supposed that $H(\xi, 0)$ should be zero (although, for $0 \leqslant \beta < 1$, $H(\xi,\beta)$ does not have the same interpretation that it does for $1 < \beta < 2$); however, for continuity of $H(\xi,\beta)$, here, it is taken that $H_0(\beta) = -(2-\beta)$, $H_{01}(\beta) = 0, \ldots$ over the whole domain $0 \leqslant \beta < 2$. In the following section, it is shown that this assumption leads to agreement with the Couette flow experimental data.

4. Results

In this section, the solutions presented in the foregoing analysis are compared with available experimental data. Because of the difficulty in the realization of plane Couette–Poiseuille flows, the amount of experimental data is meagre. A number of experiments have been performed for plane Couette flow ($\beta = 0$) and for plane Poiseuille flow ($\beta \to 2$), but, for general flows ($0 < \beta < 2$), only one source of data, for $\beta \simeq 1$, is available. In the following, comparisons are presented for these three cases, i.e. $\beta = 0$, $\beta \to 2$, and $\beta \simeq 1$.

4.1. *Plane Couette flow*

The only experimentally determined eddy viscosity data available for plane Couette flow ($\beta = 0$) are those of Reichardt (1959). These data show that $Q(y;\,0) = 1$ (cf. (2.16)) is an excellent approximation when the parameter $S(\xi, 0) = \tilde{u}_\tau(\xi, 0)\,H(\xi, 0)$ is taken to be zero. However, in § 3.5, it is shown that $H(\xi, 0) \simeq H_0(0) = -2$ for a continuous asymptotic solution for all β. It is for this reason that the term in the denominator of (2.16) is included, such that $Q(y; 0) = [1 + 2y(1-y)]^{-1}$, $H_0(0) = -2$ is the equivalent description of $Q(y;\,0) = 1$, $H_0(0) = 0$. Based in this way on Reichardt's eddy-viscosity data, the friction coefficient (cf. (3.33) and (3.34)),

$$C_f(R,0) \cong \frac{\kappa^2}{2\log^2(\kappa^2 R)}\left[1 + 2\,\frac{\log\log(\kappa^2 R) - (J_0 - \log 2)}{\log(\kappa^2 R)} + \ldots\right], \tag{4.1}$$

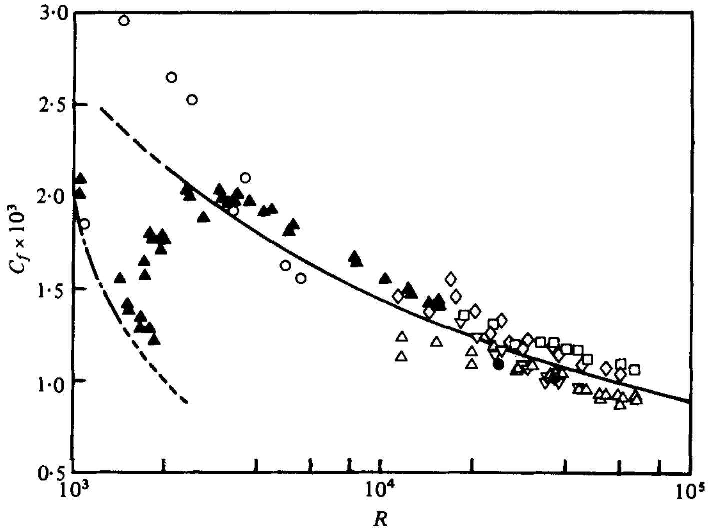

FIGURE 3. Couette flow friction coefficient *versus* Reynolds number. ——, calculation based on (4.1); —·—, $C_f = 2/R$ (laminar flow); ▲, Couette (1890), circular flow; △, ▽, Reichardt (1959), air and water; □, Robertson (1959); ◇, Robertson & Johnson (1970); ●, Chue & McDonald (1970); ○, Leutheusser & Chu (1971).

is compared to the available data in figure 3. There is reasonable overall agreement between the data and the asymptotic theory. There is some discrepancy between the data of Reichardt (1959) and Robertson (1959), and some discussion has appeared in the literature as to which set of data represents the correct friction coefficient. The present analysis finds the friction coefficient to lie between the two sets of data, when the theoretical results are based on $\kappa = 0{\cdot}41$ and $J_0 = 2{\cdot}95$, which agree with plane duct measurements (Hussain & Reynolds 1975). The low-Reynolds-number data of Leutheusser & Chu (1971) are not in good agreement with the asymptotic theory, and this discrepancy remains even when higher-order terms are included in (4.1). However, as is shown in figure 4, a relatively small error in the measured Reynolds number could be the source of this discrepancy. Figure 4 shows that the asymptotic theory covers the entire turbulent regime, from the strong variation of C_f for moderate Reynolds numbers to the weak logarithmic decay to zero for large Reynolds numbers.

It must be noted that the majority of the data points in figures 3 and 4 were not obtained from direct shear stress measurements. Rather, these points were inferred from velocity profile measurements in combination with a theory relating the slope of the velocity profile and the shear stress at the wall. Figure 5 compares the velocity measurements of Reichardt (1959) with the composite expansion (cf. (3.38)),

$$u^{(c)}(y;\xi,0) \simeq \tfrac{1}{2} + \tilde{u}_\tau(\xi,0)\left(\left[\log\left\{\frac{1+\xi y}{1+\xi(1-y)}\right\} + \{I_0(\xi y) - I_0(\xi(1-y))\}\right] + \ldots\right). \quad (4.2)$$

The thinning of the wall layers and the flattening of the core region velocity profile ($u^{(c)}(y;\xi,0) \to \frac{1}{2}$ as $\tilde{R} \to \infty$) are evident as the Reynolds number increases. Reichardt (1959) found it necessary to introduce two analytical models to fit the low-and-high Reynolds-number data, and concluded that his data at $R = 11\,800$ were in error

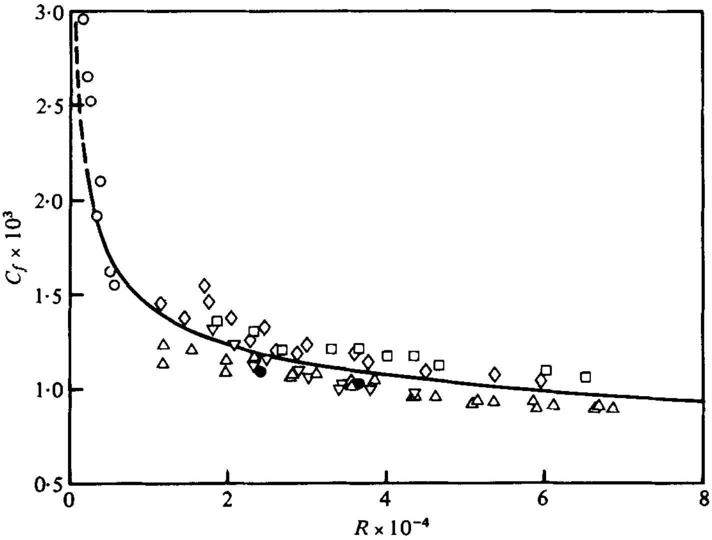

FIGURE 4. Couette flow friction coefficient *versus* Reynolds number. Symbols as in figure 3.

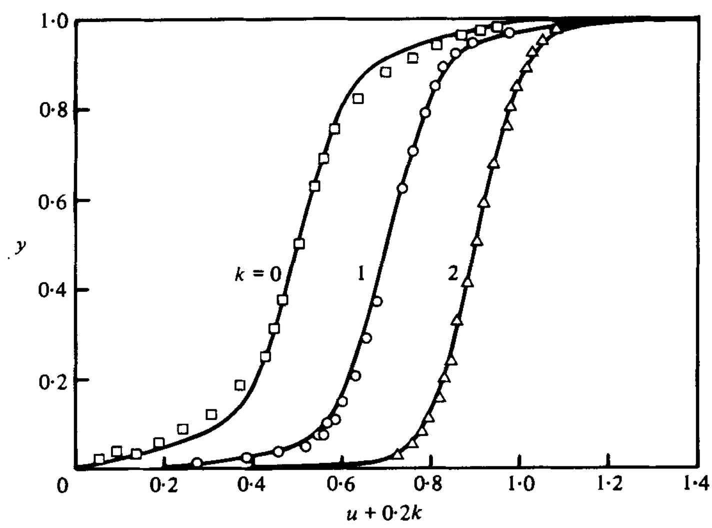

FIGURE 5. Couette flow velocity profiles. ——, calculation based on (4.2); □, ○, △, data of Reichardt (1959) for R = 5800, 11800 and 68000, respectively.

because of lack of agreement with the models. In fact, these measurements are in good agreement with the present asymptotic theory results, as is shown in figure 4 (some of the measurements in oil at $R = 5800$ are noti n as good agreement). The velocity measurements of Robertson & Johnson (1970) are also in good agreement with the results of the present theory, at least in the overlap region, as is shown in figure 6.†

† The data of Robertson & Johnson (1970) are inaccurate very near the wall, as these authors indicate. However, in the overlap domain, where their measurements are more accurate, the results of the present theory for the wall regions, based on the Van Driest exponential-decay

Thus, the discrepancy between the friction coefficients of these two sets of data appears to be due not so much to errors in velocity measurements, but more to the inaccurate deduction of C_f from the measurements.

4.2. *Plane Poiseuille flow*

In §§ 3.4 and 3.5, it is shown that the maximum velocity for $1 < \beta < 2$ is given by $U(\beta) = 1/(2-\beta)$. Hence, the $\wedge$-superscripted variables in § 3.4 refer to Couette–Poiseuille flow scaled on the maximum velocity, instead of the moving wall velocity. In particular, the theoretical Poiseuille flow friction coefficient, given by

$$\hat{C}_f(\hat{R}, 2) = 2\{\kappa\hat{u}_\tau(\hat{R}, 2)\}^2 \cong \frac{2\kappa^2}{\log^2 \hat{R}}\left[1 + 2\frac{\log\log\hat{R} - \hat{E}_0(2)}{\log\hat{R}} + \dots\right], \tag{4.3}$$

is compared in figure 7 with the measurements of Hussain & Reynolds (1975). Based on this comparison, it is determined that

$$\hat{E}_0(2) = (J_0 - 1) + \hat{\Upsilon}_0(2) \doteqdot 2\cdot 60 \tag{4.4a}$$

$$\Rightarrow \hat{\Upsilon}_0(2) = \hat{E}_0(2) - (J_0 - 1) \doteqdot 0\cdot 65. \tag{4.4b}$$

As stated in the introduction, plane Poiseuille flow is the limiting case for the Couette–Poiseuille flow solutions presented in § 3. To investigate this limit, consider $\alpha = (2-\beta)$ and $\delta(\alpha) = (y_d(\beta) - \frac{1}{2})$ as $\alpha, \delta(\alpha) \to 0$. In this limit, (3.44*a*) yields

$$\delta(\alpha) = \tfrac{1}{4}[1 - \tfrac{1}{2}Q(\tfrac{1}{2};\, 2)]\,\alpha[1 + O(\alpha)], \tag{4.5}$$

where $Q(\frac{1}{2};\, 2) \doteqdot 0\cdot 30$ (cf., e.g., Hussain & Reynolds 1975); while (3.44*b*) yields

$$\hat{\Upsilon}_0(2) - \Theta_0(\tfrac{1}{2};\, 2) = -(\log 4 - 1), \quad \dots \tag{4.6a}$$

$$\Rightarrow \Theta_0(\tfrac{1}{2};\, 2) = \hat{\Upsilon}_0(2) + (\log 4 - 1) \doteqdot 1\cdot 04. \tag{4.6b}$$

Thus, (4.5) indicates that the defect distance approaches the value $\frac{1}{2}$ linearly as the plane Poiseuille flow distribution is approached. It is noted that the zero Reynolds stress location is $y_n(\beta) = 1/\beta$, such that, in this limit,

$$\{y_n(\beta) - y_d(\beta)\} = \tfrac{1}{8}Q(\tfrac{1}{2}; 2)\,(2-\beta)\,[1 + O(2-\beta)]. \tag{4.7}$$

Further, (4.4*b*) and (4.6*b*) provide the following integral conditions that $D(y;\, 2)$ must satisfy (cf. (3.5*c*)):

$$\int_0^1 \left[\frac{(1-2y)}{y(1-y)}\right] D(y;\, 2)\, dy = 0, \quad \int_0^1 \left[\frac{1}{(1-y)}\right] D(y;\, 2)\, dy = \hat{\Upsilon}_0(2) \doteqdot 0\cdot 65; \tag{4.8a}$$

$$\int_0^{\frac{1}{2}} \left[\frac{(1-2y)}{y(1-y)}\right] D(y;\, 2)\, dy = \Theta_0(\tfrac{1}{2};\, 2) \doteqdot 1\cdot 04. \tag{4.8b}$$

It is noted that the first integral of (4.8*a*) implies that $D(y;\, 2)$ is symmetric about $y = \frac{1}{2}$.

model for the damping function, are in good agreement with these experimental results. Further, the theoretical results for the wall regions, based on this damping function model, are in *excellent* agreement with the experimental results of Hussain & Reynolds (1975), which incorporate more accurate near-wall measurements. Based on these comparisons, this exponential-decay model was employed throughout the present theoretical analysis, and the surface-renewal or algebraic-decay model was not considered.

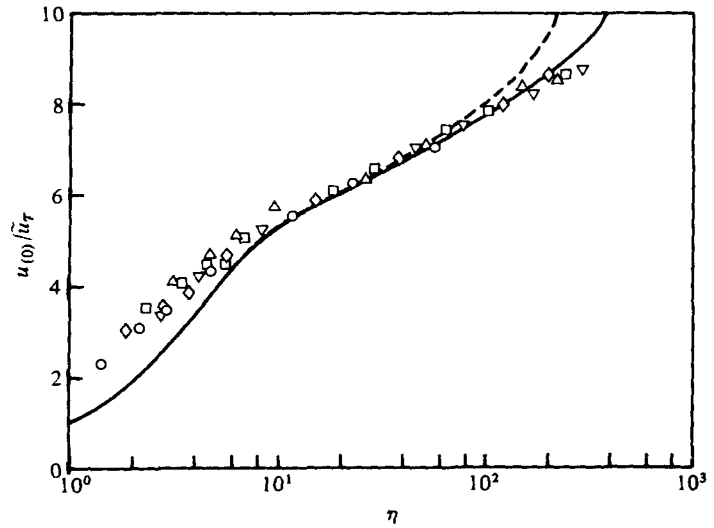

FIGURE 6. Couette flow velocity profile in wall variables. —— and – – –, calculations based on (4.2) for $R = 66000$ and 28200, respectively; ○, ◇, □, ▽, △, data of Robertson & Johnson (1970) for $R = 28200$, 37640, 47200, 56400 and 66000, respectively.

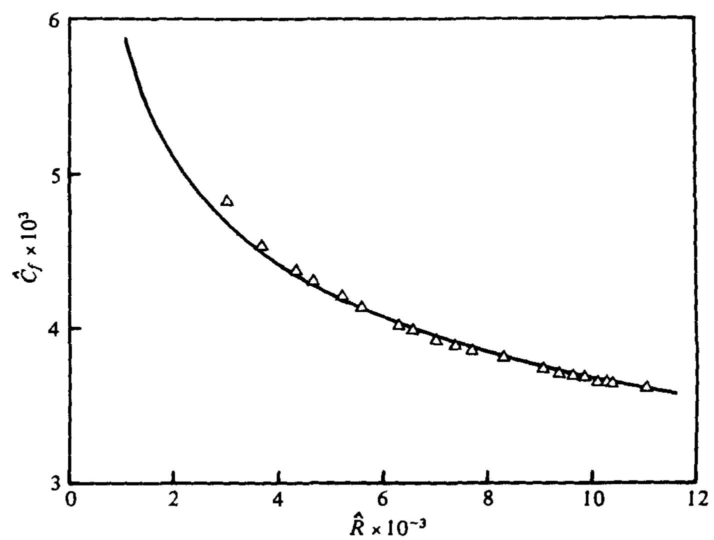

FIGURE 7. Poiseuille flow friction coefficient *versus* Reynolds number. ——, calculation based on (4.3); △, data of Hussain & Reynolds (1975).

4.3. *Zero net flow*

For $\beta \neq 0, 2$, the only source of experimental data available is for the zero net flow experiment (cf. Huey & Williamson 1974). In this experiment, the ends of the plane duct with one moving wall are blocked, so that no flow can enter or leave the duct. This blocking results in an adverse pressure gradient, causing flow reversal near the stationary wall. Translation of the variables, i.e. $y'' = (1-y)$, $u'' = (1-u)$, gives the zero net flow condition, in terms of the variables of the present formulation, as

$$\int_0^1 u(y;\tilde{R},\beta)\,dy = 1. \tag{4.9}$$

With the approximation that

$$u(y;\tilde{R},\beta) = u^{(c)}(y;\xi,\beta) \tag{4.10}$$

where the function $u^{(c)}$ is given in (3.38), the condition of (4.9) becomes

$$(\beta-1) \cong \tilde{u}_\tau(\xi,\beta)\,([(2-\beta)\{\tfrac{1}{6}\beta(5-\beta)+\Lambda_0(\beta)\}+(\beta-1)\,\Psi_0(\beta)]+\ldots), \tag{4.11a}$$

where

$$\Lambda_0(\beta) = \int_0^1 \left[\frac{(1-\beta t)}{(1-t)}\right] D(t;\beta)\,dt. \tag{4.11b}$$

Because of (4.9), β must change as the (translated) moving wall Reynolds number changes, i.e. $\beta = \beta(\tilde{R})$. Consider that

$$\beta(\tilde{R}) = 1+\alpha(\tilde{R}), \quad \text{with} \quad \alpha(\tilde{R}) \to 0 \quad \text{as} \quad \tilde{R}\to\infty, \tag{4.12a}$$

such that

$$\tilde{u}_\tau(\tilde{R},\beta(\tilde{R})) \cong \tilde{u}_\tau(\tilde{R},1)\,[1+\alpha(\tilde{R})+\ldots],$$

with

$$\tilde{u}_\tau(\tilde{R},1) \cong \frac{1}{\log\tilde{R}}\left[1+\frac{\log\log\tilde{R}-\{(J_0-\tfrac{1}{2})+\Psi_0(1)\}}{\log\tilde{R}}+\ldots\right]. \tag{4.12b}$$

Substitution of (4.12a,b) into (4.11) yields

$$\alpha(\tilde{R}) = \{\tfrac{2}{3}+\Lambda_0(1)\}\,\tilde{u}_\tau(\tilde{R},1)\,[1+O(\tilde{u}_\tau(\tilde{R},1))]. \tag{4.13}$$

Based on these results, the friction coefficient is given by

$$\begin{aligned} C_f(R) &= C_f(\tilde{R}) = \{2\kappa\tilde{u}_\tau(\tilde{R})\}^2 \\ &\cong \frac{2\kappa^2}{\log^2\tilde{R}}\left[1+2\,\frac{\log\log\tilde{R}-\{(J_0-\tfrac{7}{6})+\Psi_0(1)-\Lambda_0(1)\}}{\log\tilde{R}}+\ldots\right]. \end{aligned} \tag{4.14}$$

Huey & Williamson (1974) present pressure gradient measurements in the form

$$\begin{aligned} P''(R) &\equiv \frac{|(dp^*/dx^*)|}{(\rho^*\nu^*V^*/h^{*2})} = R\,P(R,\beta(R)) = \tilde{R}\,\beta(\tilde{R})\,\tilde{u}_\tau^2(\tilde{R},\beta(\tilde{R})) \\ &\cong \frac{\tilde{R}}{\log^2\tilde{R}}\left[1+2\,\frac{\log\log\tilde{R}-\{(J_0-\tfrac{3}{2})+\Psi_0(1)-\tfrac{3}{2}\Lambda_0(1)\}}{\log\tilde{R}}+\ldots\right]. \end{aligned} \tag{4.15}$$

For the case of zero net flow, the maximum velocity (cf. § 3.5) can be expressed as

$$\begin{aligned} u_m(R) = u_m(\tilde{R}) &= \frac{1}{2-\beta(\tilde{R})} = \frac{1}{1-\alpha(\tilde{R})} \\ &= 1+\alpha(\tilde{R})\,[1+O(\alpha(\tilde{R}))] \\ &\cong 1+\frac{\{\tfrac{2}{3}+\Lambda_0(1)\}}{\log\tilde{R}}\,[1+\ldots]. \end{aligned} \tag{4.16}$$

A comparison of (4.15) with the experimental measurements of $P''(Re)$ gives

$$G_0(1) = \{(J_0-\tfrac{3}{2})+\Psi_0(1)-\tfrac{3}{2}\Lambda_0(1)\} \doteqdot 2{\cdot}7 \tag{4.17a}$$

$$\Rightarrow \{\Psi_0(1)-\tfrac{3}{2}\Lambda_0(1)\} = G_0(1)-(J_0-\tfrac{3}{2}) \doteqdot 1{\cdot}2. \tag{4.17b}$$

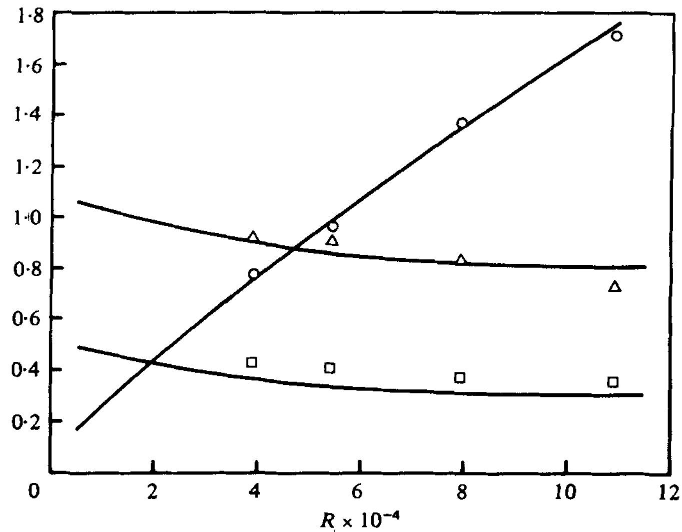

FIGURE 8. Comparison of asymptotic theory with Huey & Williamson's (1974) data for the zero net flow problem. ——, theory; △, (translated) maximum velocity measurement, $(u_m - 1) \times 10$; ⊙, pressure gradient measurement, $P'' \times 10^{-2}$; □, friction coefficient calculation, $C_f \times 10^4$.

Further, a comparison of (4.16) with the measurements of $u_m(R)$ gives

$$\{\tfrac{2}{3} + \Lambda_0(1)\} \doteqdot 0{\cdot}78. \tag{4.18}$$

Thus, (4.17) and (4.18) provide the following integral conditions that $D(y;\,1)$ must satisfy:

$$\int_0^1 \frac{1}{y} D(y;1)\,dy = \Psi_0(1) \doteqdot 1{\cdot}4; \quad \int_0^1 D(y;\,1)\,dy = \Lambda_0(1) \doteqdot 0{\cdot}11. \tag{4.19}$$

The above comparisons for $P''(R)$ and $(u_m(R) - 1)$ are shown in figure 8. Also shown in this figure is the friction coefficient of (4.14), based upon the values of $\Psi_0(1)$ and $\Lambda_0(1)$ of (4.19), in comparison with that determined by the calculations of Huey & Williamson. These calculations of $C_f(R)$, deduced by the method of Robertson & Johnson (1970), appear slightly too high, as do the calculated values of Robertson & Johnson for $C_f(R)$ for plane Couette flow (cf. § 4.1).

Consider now the behaviour of $y_d(\beta)$ as $\beta \to 1$. If $\alpha = (\beta - 1)$ and $\delta(\alpha) = (1 - y_d(\beta))$, with $\alpha, \delta(\alpha) \to 0$, then it is determined, from (3.44*a*), that

$$\delta(\alpha) = \frac{1}{\sqrt{2}} \alpha^{\frac{1}{2}}[1 + O(\alpha^{\frac{1}{2}})]. \tag{4.20}$$

Since $y_n(\beta) = 1/\beta$, as previously noted, in the limit of $(\beta - 1) \to 0$,

$$\{y_n(\beta) - y_d(\beta)\} = \frac{1}{\sqrt{2}} (\beta - 1)^{\frac{1}{2}} [1 + O((\beta - 1)^{\frac{1}{2}})]. \tag{4.21}$$

Finally, it is noted that (3.44*b*), in this limit, yields just the identity $\Theta_0(1;\,1) = \Psi_0(1)$.

5. Conclusions

By the introduction of a suitable modification of the conventional eddy-viscosity closure model for turbulent shear flow in plane ducts, an asymptotic solution is obtained which is continuous for the complete family of Couette–Poiseuille flows ($0 \leqslant \beta < 2$) as the Reynolds number tends to infinity. It is shown that, by the inclusion of higher-order terms in the asymptotic expansion ($F_{01}(y;\beta), \ldots$), the position of zero Reynolds stress ($y_n(\beta)$) is displaced farther from the stationary wall than is the position of maximum velocity ($y_m(\beta)$) for $1 < \beta < 2$ (i.e. $H_0(\beta) = -(2-\beta) < 0$), in agreement with experimental observations. Previously, this difference in $y_n(\beta)$ from $y_m(\beta)$ has not been taken into account in the analysis of Couette–Poiseuille flows.

Through the combination of the matched asymptotic expansions of the velocity of the core, stationary wall and moving wall regions of the flow field, a composite expansion is obtained for the velocity field, which is uniformly valid across the whole duct as the Reynolds number tends to infinity. This composite expansion is found to be in excellent agreement with the available velocity measurements for plane Couette flow ($\beta = 0$). The matching requirements at each wall region lead to a skin friction law of the same asymptotic form as obtained previously for plane Poiseuille flow. This form gives a good representation of the skin friction measurements for plane Couette flow ($\beta = 0$), plane Poiseuille flow ($\beta \to 2$), and the case of zero net flow ($\beta \simeq 1$). Although the asymptotic theory cannot provide the closure equations for a given turbulent shear flow problem, it does provide certain restrictive conditions which an assumed closure model must satisfy, as is shown in this work and elsewhere.

Apart from significance in its own right, the present analysis should be useful for (i) the further development of turbulent lubrication theory, which, up to now, has relied almost entirely on numerical solutions; and (ii) guidance in the closure formulations of further numerical computations.

The work presented in this paper was supported in part by the United States Air Force Office of Scientific Research under Grant AFOSR 77-3362 and in part by the National Science Foundation under Grant ENG 77-18730. The authors wish to acknowledge the encouragement given by Dr F. A. Williams of the University of California, San Diego.

REFERENCES

Bush, W. B. & Fendell, F. E. 1972 Asymptotic analysis of turbulent channel and boundary-layer flow. *J. Fluid Mech.* **56**, 657–681.

Bush, W. B. & Fendell, F. E. 1973 Asymptotic analysis of turbulent channel flow for mean turbulent energy closures. *Phys. Fluids* **16**, 1189–1197.

Bush, W. B. & Fendell, F. E. 1974 Asymptotic analysis of turbulent channel flow for mixing length theory. *SIAM J. Appl. Math.* **26**, 413–427.

Burgers, J. M. 1922 Ein Versuch zur Abschätzung des turbulenten Strömungswiderstandes. In *Vorträge aus dem Gebiete der Hydro- und Aerodynamic* (ed. T. von Kármán & T. Levi-Civita), pp. 143–145. Springer.

Chue, S. H. 1969 An experimental investigation on turbulent plane Couette flow in dilute polymer solutions. MS thesis, Purdue University.

Chue, S. H. & McDonald, A. T. 1970 Application of a new effective viscosity model to turbulent plane Couette flow. *A.I.A.A. J.* **8**, 2076–2078.

COLE, J. D. 1968 *Perturbation Methods in Applied Mathematics*. Watham, Massachusetts: Blaisdell.

CONSTANTINESCU, V. N. 1959 On turbulence lubrication. *Proc. Inst. Mech. Eng.* **173**, 881–900*d*.

COUETTE, M. 1890 Etudes sur le frottement des liquides. *Ann. Chem. Phys.* **21**, (6) 433–510.

ELROD, H. G. & NG, C. W. 1967 A theory for turbulent fluid films and its application to bearings. *J. Lubr. Tech.* **89**, 346–362.

FENDELL, F. E. 1972 Singular perturbation and turbulent shear layers near walls. *J. Astro. Sci.* **20**, 129–165.

HANJALIĆ, K. & LAUNDER, B. E. 1972 Fully developed asymmetric flow in a plane channel. *J. Fluid Mech.* **51**, 301–335.

HEISENBERG, W. 1922 Nichtlaminare Lösungen der Differentialgleichungen für reibende Flüssigkeiten. In *Vorträge aus dem Gebiete der Hydro- und Aerodynamic* (ed. T. von Kármán & T. Levi-Civita), pp. 139–142. Springer.

HOFFMEISTER, M. 1976 Über eine lokale Abschlussbedingung für turbulente Scherströmungen. *Z. angew. Math. Mech.* **56**, 415–419.

HO, M. K. & VOHR, J. H. 1974 Application of energy model of turbulence to calculation of lubricant flows. *J. Lubr. Tech.* **96**, 95–102.

HUEY, L. J. & WILLIAMSON, J. W. 1974 Plane turbulent Couette flow with zero net flow. *J. Appl. Mech.* **41**, 885–890.

HUSSAIN, A. K. M. F. & REYNOLDS, W. C. 1975 Measurements in fully developed turbulent channel flow. *J. Fluids Engng* **97**, 568–580.

KÁRMÁN, T. VON 1937 The fundamentals of the statistical theory of turbulence. *J. Aero. Sci.* **4**, 131–138.

KORKEGI, R. H. & BRIGGS, R. A. 1968 On compressible turbulent plane Couette flow. *A.I.A.A. J.* **6**, 742–744.

KORKEGI, R. H. & BRIGGS, R. A. 1970 Compressible turbulent plane Couette flow with variable heat transfer based on von Kármán model. *A.I.A.A. J.* **8**, 817–819.

LEUTHEUSSER, H. J. & CHU, V. H. 1971 Experiments on plane Couette flow. *Proc. A.S.C.E. J. Hydraul. Div.* **97**, 1269–1284.

NG, C. W. 1964 Fluid dynamic foundation of turbulent lubrication theory. *Trans. A.S.L.E.* **7**, 311–321.

REHME, K. 1974 Turbulent flow in smooth concentric annuli with small radius ratios. *J. Fluid Mech.* **64**, 263–287.

REICHARDT, H. 1956 Über die Geschwindigkeitsverteilung in einer geradlinigen turbulenten Couetteströmung. *Z. angew. Math. Mech. Sonderheft* **36**, S26–29.

REICHARDT, H. 1959 Gesetzmässigkeiten der geradlinigen turbulenten Couetteströmung. *Mitt. Max-Planck-Inst. fur Strömungsforschung*, no. 22, Göttingen.

REYNOLDS, W. C. 1976 Computation of turbulent flows. *Ann. Rev. Fluid Mech.* **8**, 183–208.

ROBERTSON, J. M. 1959 On turbulent plane Couette flow. *Proc. 6th Midwestern Conf. on Fluid Mech.*, pp. 169–182. University of Texas, Austin.

ROBERTSON, J. M. & JOHNSON, H. F. 1970 Turbulence structure in plane Couette flow. *Proc. A.S.C.E., J. Eng. Mech. Div.* **96**, 1171–1182.

SZABLEWSKI, W. 1968 Turbulente Parallelströmungen. *Z. angew. Math. Mech.* **48**, 35–50.

WHITE, F. M. 1974 *Viscous Fluid Flow*. McGraw-Hill.

YAJNIK, K. S. 1970 Asymptotic theory of turbulent shear flows. *J. Fluid Mech.* **42**, 411–427.